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TENSOR PRODUCT OF DIFFERENCE POSETS

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ABSTRACT. A tensor product of difference posets, which generalize orthoalgebras and orthomodular posets, is defined, and an equivalent condition is presented. In particular, we show that a tensor product for difference posets with a sufficient system of probability measures exists, as well as a tensor product of any difference poset and any Boolean algebra, which is isomorphic to a bounded Boolean power.

1. INTRODUCTION

In the axiomatic approach to quantum mechanics, the event structure of a physical system is identified with a quantum logic [2] or an orthoalgebra [20, 8] versus with a Boolean algebra in the case of a classical mechanics [18]. Assume that we have two independent physical systems with event structures P and Q and wish to regard them as a coupled system. The event structure L of this coupled system is usually called a tensor product of P and Q, and we write $L = P \otimes Q$.

Tensor products in various approaches have been studied in [21, 1, 16, 6, 10, 12, 15, 19, 20, 22, 23]. A tensor product of orthoalgebras has been investigated by Foulis and Bennett in [7] via a universal mapping property, and a tensor product of an orthoalgebra and a Boolean algebra is given in [9].

Recently there has appeared a new axiomatic model, *difference posets* (or D-posets, for short), introduced by Kôpka and Chovanec [14], which generalize quantum logics, orthoalgebras as well as the set of all effects (i.e., the system of all Hermitian operators A on a Hilbert space H with $O \leq A \leq I$, which are important for modeling Hilbert space quantum mechanics). Difference posets have been inspired with a possibility to introduce fuzzy set ideas to quantum structure models [2]. In this model, a difference operation is a primary notion from which it is possible to derive other usual notions important for measurements.

The aim of the present paper is to introduce a tensor product for difference posets via a universal mapping property. We show how to construct such a tensor product for difference posets with a sufficient system of probability mea-

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sures. In particular, we show that the tensor product of a difference poset and a Boolean algebra always exists and is isomorphic to a bounded Boolean power. We also give an example when the tensor product of orthoalgebras fails as an orthoalgebra, while it exists in the class of difference posets.

2. Difference posets

A D-*poset*, or a *difference poset*, is a partially ordered set L with a partial ordering $\leq$, greatest element 1, and with a partial binary operation $\ominus: L\times L \to L$, called a *difference*, such that, for $a, b \in L$, $b \ominus a$ is defined if and only if $a \leq b$, and such that the following axioms hold for $a, b, c \in L$:

(DPi) $b \ominus a \leq b$;

(DPii) $b \ominus (b \ominus a) = a$;

(DPiii) $a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The following statements have been proved in [14]:

Proposition 2.1. *Let a, b, c, d be elements of a* D-*poset* L. *Then*

(i) $1 \ominus 1$ *is the smallest element of* L; *denote it by* 0.

(ii) $a \ominus 0 = a$.

(iii) $a \ominus a = 0$.

(iv) $a \leq b \Rightarrow b \ominus a = 0 \Leftrightarrow b = a$.

(v) $a \leq b \Rightarrow b \ominus a = b \Leftrightarrow a = 0$.

(vi) $a \leq b \leq c \Rightarrow b \ominus a \leq c \ominus a$ *and* $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.

(vii) $b \leq c,\ a \leq c \ominus b \Rightarrow b \leq c \ominus a$ *and* $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.

(viii) $a \leq b \leq c \Rightarrow a \leq c \ominus (b \ominus a)$ *and* $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

Remark 2.2 ([17]). A poset L with smallest and greatest elements 0 and 1, respectively, and with a partial binary operation $\ominus: L \times L \to L$ such that for $a, b, c \in L$ we have

(i) $a \ominus 0 = a$;

(ii) if $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$,

is a D-poset.

For any element $a \in L$ we put

$$a^{\perp} := 1 \ominus a.$$

Then (i) $a^{\perp\perp} = a$; (ii) $a \leq b$ implies $b^{\perp} \leq a^{\perp}$. Two elements a and b of L are *orthogonal*, and we write $a \perp b$, iff $a \leq b^{\perp}$ (iff $b \leq a^{\perp}$).

Now we introduce a binary operation $\oplus: L \times L \to L$ such that an element $c = a \oplus b$ in L is defined iff $a \perp b$, and for c we have $b \leq c$ and $a = c \ominus b$. The partial operation $\oplus$ is defined correctly because if there exists $c_1 \in L$ with $b \leq c_1$ and $a = c_1 \ominus b$, then, by Proposition 2.1(viii) and (DPii), we have

$$(1 \ominus (c \ominus b)) \ominus b = 1 \ominus c = (1 \ominus (c_1 \ominus b)) \ominus b = 1 \ominus c_1,$$

which implies $c = c_1$. Moreover, by [5],

$$c = a \oplus b = (a^{\perp} \ominus b)^{\perp} = (b^{\perp} \ominus a)^{\perp}. \tag{2.1}$$

The operation $\oplus$ is commutative (this is evident) and associative: suppose that $y = a \oplus b$ and $z = (a \oplus b) \oplus c$ exist in L. By (DPiii) we have

$$(z \ominus a) \ominus (z \ominus y) = y \ominus a, \quad (z \ominus a) \ominus c = b,$$
$$z \ominus a = b \oplus c \in L, \quad z = a \oplus (b \oplus c) \in L.$$

Very important examples of difference posets are orthomodular posets (= quantum logics), orthoalgebras, and sets of effects.[1]

3. Orthomodular posets

An *orthomodular poset* (OMP) is a partially ordered set L with an ordering $\leq$, the smallest and greatest elements 0 and 1, respectively, and an orthocomplementation $\perp : L \to L$ such that

(OMi) $a^{\perp\perp} = a$ for any $a \in L$;

(OMii) $a \vee a^{\perp} = 1$ for any $a \in L$;

(OMiii) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$;

(OMiv) if $a \leq b^{\perp}$ (and we write $a \perp b$), then $a \vee b \in L$;

(OMv) if $a \leq b$, then $b = a \vee (a \vee b^{\perp})^{\perp}$ (orthomodular law).

If in an orthomodular poset L the join of any sequence (any system) of mutually orthogonal elements exists, we say that L is a *σ-orthomodular poset* (a *complete orthomodular poset*). An *orthomodular lattice* is an orthomodular poset L such that, for any $a, b \in L$, $a \vee b$ exists in L (using de Morgan laws, $a \wedge b$ exists in L, too). A distributive orthomodular lattice is called a *Boolean algebra.* We recall that an orthomodular lattice L is a Boolean algebra iff for any pair $a, b \in L$ there are three mutually orthogonal elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. For more details concerning orthomodular posets and lattices see, for example, [11, 18].

One of the most important cases of orthomodular lattices is the system of all closed subspaces, $L(H)$, of a real or complex Hilbert space H, with an inner product $(\cdot, \cdot)$. Here the partial ordering, $\leq$, is induced by the natural set-theoretic inclusion, and $M^{\perp} = \{x \in H\colon (x, y) = 0 \text{ for any } y \in M\}$. Then $L(H)$ is a complete orthomodular lattice, which is not a Boolean algebra, if $\dim H \neq 1$. This structure plays a crucial role in axiomatic foundations of quantum mechanics.

If for two elements a, b of an OMP L, with $a \leq b$, we define by (OMv)

$$b \ominus a := (a \vee b^{\perp})^{\perp},$$

then L with $\leq$, 1, and $\ominus$ is a difference poset.

4. Orthoalgebras

An *orthoalgebra* is a set L with two particular elements 0, 1 and with a partial binary operation $\oplus :\ L \times L \to L$ such that for all $a, b, c \in L$ we have

(OAi) if $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity);

(OAii) if $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity);

(OAiii) for any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation);

(OAiv) if $a \oplus a$ is defined, then $a = 0$ (consistency).

If the assumptions of (OAii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L.

Let a and b be two elements of an orthoalgebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less than*

[1] Any D-poset L can be regarded as a Brower–Zadeh (BZ)-poset $(L, 0, \leq, ', {}^{\sim})$ introduced by Cattaneo and Nisticò [3], when we put $a' := a^{\perp}$, and $a^{\sim} := 1$ iff $a = 0$ and $a^{\sim} := 0$ iff $a \neq 0$. In that framework, fuzzy sets and effects are studied, too.

or equal to b and write $a \leq b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \geq a$); (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$ and it is written as $a^\perp$.

If $a \leq b$, for the element c in (ii) with $a \oplus c = b$ we write $c = b \ominus a$, and c is called the *difference* of a and b. It is evident that

$$b \ominus a = (a \oplus b^\perp)^\perp. \tag{4.1}$$

In [8], there are proofs of the main properties of orthoalgebras.

We note that if L is an orthomodular poset and $a \oplus b := a \vee b$ whenever $a \perp b$ in L, then L with $0, 1, \oplus$ is an orthoalgebra. The converse statement does not hold, in general. We recall that an orthoalgebra L is an OMP iff $a \perp b$ implies $a \vee b \in L$.

It is evident that any orthoalgebra L is a D-poset when a difference $\ominus$ is defined by (4.1). Indeed, (DPi) and (DPii) are trivially satisfied, and (DPiii) follows from (xix) of Proposition 4.1 in [5].

By [17], we conclude that a D-poset L with $0, 1$, and $\oplus$, defined by (2.1), is an orthoalgebra if and only if $a \leq 1 \ominus a$ implies $a = 0$. Therefore, it is not hard to give many examples of D-posets which are not orthoalgebras; for instance, sets of effects:

Example 4.1. *The set $\mathscr{E}(H)$ of all Hermitian operators A on H such that $O \leq A \leq I$, where I is the identity operator on H, is a difference poset which is not an orthoalgebra; a partial ordering $\leq$ is defined via $A \leq B$ iff $(Ax, x) \leq (Bx, x)$, $x \in H$, and $C = B \ominus A$ iff $(Ax, x) - (Bx, x) = (Cx, x)$, $x \in H$.*

This set plays an important role for unsharp measurements of quantum mechanics, [2].

5. $\bigoplus$-ORTHOGONAL SYSTEMS

Let $F = \{a_1, \dots, a_n\}$ be a finite sequence in L. Recursively we define for $n \geq 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n, \tag{5.1}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in D-posets we conclude that (5.1) is correctly defined. By definition we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$ and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$ and any k with $1 \leq k \leq n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n}, \tag{5.2}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n). \tag{5.3}$$

We say that a finite sequence $F = \{a_1, \dots, a_n\}$ in L is *$\bigoplus$-orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case we say that F has a *$\bigoplus$-sum*, $\bigoplus_{i=1}^n a_i$, defined via

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus \cdots \oplus a_n. \tag{5.4}$$

It is clear that two elements a and b of L are orthogonal, i.e. $a \perp b$, iff $\{a, b\}$ is $\bigoplus$-orthogonal.

An arbitrary system $G = \{a_i : i \in I\}$ of not necessarily different elements of L is $\bigoplus$-*orthogonal* iff, for every finite subset F of I, the system $\{a_i : i \in F\}$ is $\bigoplus$-orthogonal. If $G = \{a_i : i \in I\}$ is $\bigoplus$-orthogonal, so is any $\{a_i : i \in J\}$ for any $J \subseteq I$. A $\bigoplus$-orthogonal system $G = \{a_i : i \in I\}$ of L has a $\bigoplus$-*sum* in L, written as $\bigoplus_{i\in I} a_i$, iff in L there exists the join

$$\bigoplus_{i\in I} a_i := \bigvee_F \bigoplus_{i\in F} a_i, \tag{5.5}$$

where F runs over all finite subsets in I. In this case, we also write $\bigoplus G := \bigoplus_{i\in I} a_i$.

It is evident that if $G = \{a_1, \dots, a_n\}$ is $\bigoplus$-orthogonal, then the $\bigoplus$-sums defined by (5.4) and (5.5) coincide.

We say that a D-poset L is a *complete D-poset* (*σ-D-poset*) if, for any $\bigoplus$-orthogonal system (any $\bigoplus$-orthogonal sequence) G of L, the $\bigoplus$-sum exists in L. It is straightforward to verify that a D-poset L is a D-σ-poset if, for any sequence $\{a_i\}$ in L with $a_1 \le a_2 \le \cdots$, the join $\bigvee_{i=1}^{\infty} a_i$ exists in L.

A finite *decomposition* of 1 is any $\bigoplus$-orthogonal finite sequence $\{a_1, \dots, a_n\}$ such that $\bigoplus_{i=1}^{n} a_i = 1$.

Proposition 5.1. *Let $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_m\}$ be $\bigoplus$-orthogonal systems of elements in a difference poset L. Then $\{a_1, \dots, a_n, b_1, \dots, b_m\}$ is $\bigoplus$-orthogonal iff $a := \bigoplus_{i=1}^{n} a_i \perp b := \bigoplus_{j=1}^{m} b_j$. Then*

$$\bigoplus\{a_1, \dots, a_n, b_1, \dots, b_m\} = a \oplus b. \tag{5.6}$$

Proof. It follows directly from the definition of the $\bigoplus$-orthogonality. □

We recall that for orthoalgebras and orthomodular posets we have $\bigoplus A \perp \bigoplus B$, iff $A \cap B \subseteq \{0\}$, and $A \cup B$ is a $\bigoplus$-orthogonal set, but for D-posets it is not true. For example, for the difference poset $\mathscr{E}(H)$, $A = \{\frac{1}{2}I\}$ and $B = \{\frac{1}{2}I\}$ are $\bigoplus$-orthogonal and $\{\frac{1}{2}I, \frac{1}{2}I\}$ is a finite decomposition of I, but $A \cap B \neq \{0\}$.

6. Probability measures and morphisms of difference posets

Let L be a D-poset. A mapping $\mu : L \to [0, 1]$ such that $\mu(1) = 1$, and $\mu(a \oplus b) = \mu(a) + \mu(b)$, $a, b \in L$, is said to be a *probability measure* (or also a *state*) on L. Denote by $\Omega(L)$ the set of all probability measures on L. It is well known [11, 18] that there are examples of nontrivial orthomodular lattices and OMPs such that $\Omega(L) = \varnothing$ or $\Omega(L)$ is a singleton. We say that $\Omega(L)$ is *sufficient* iff, for any nonzero $a \in L$, there is $\mu \in \Omega(L)$ such that $\mu(a) \neq 0$.

Let P and L be two D-posets. A mapping $\phi : P \to L$ is said to be

(i) a *morphism* iff $\phi(1) = 1$, and $p \perp q$, $p, q \in P$, implies $\phi(p) \perp \phi(q)$ and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$;
(ii) a *monomorphism* iff ϕ is a morphism and $\phi(p) \perp \phi(q)$ iff $p \perp q$;
(iii) an *isomorphism* iff ϕ is a surjective monomorphism.

Let P, Q, L be D-posets. A mapping $\beta : P \times Q \to L$ is called a *bimorphism* iff

(i) $a, b \in P$ with $a \perp b$, $q \in Q$ imply $\beta(a, q) \perp \beta(b, q)$ and $\beta(a \oplus b, q) = \beta(a, q) \oplus \beta(b, q)$;

(ii) $c, d \in Q$ with $c \perp d$, $p \in P$ imply $\beta(p, c) \perp \beta(p, d)$ and $\beta(p, c \oplus d) = \beta(p, c) \oplus \beta(p, d)$;

(iii) $\beta(1, 1) = 1$.

If $\beta : P \times Q \to L$ is a bimorphism, then $\beta(\cdot, 1) : P \to L$ and $\beta(1, \cdot) : Q \to L$ are morphisms. Therefore, for $p \in P$ and $q \in Q$, we have $\beta(p, 1)^\perp = \beta(p^\perp, 1)$, $\beta(1, q)^\perp = \beta(1, q^\perp)$, and $\beta(p, 0) = \beta(0, q) = 0$.

Also, if $a, b, p \in P$ and $c, d, q \in Q$, we have $a \leq b \Rightarrow \beta(a, q) \leq \beta(b, q)$ and $c \leq d \Rightarrow \beta(p, c) \leq \beta(p, d)$.

Example 6.1. *Let H_1 and H_2 be two Hilbert spaces over the same field, and let $H = H_1 \otimes H_2$ be their usual tensor product. The mapping $\beta : \mathscr{E}(H_1) \otimes \mathscr{E}(H_2) \to \mathscr{E}(H)$ defined via $\beta(A, B) = A \otimes B$, $A \in \mathscr{E}(H_1)$, $B \in \mathscr{E}(H_2)$, is a bimorphism.*

If $K = \{(p_i, q_i)\}_{i=1}^n$, $p_i \in P$, $q_i \in Q$, $i = 1, \dots, n$, we define $\beta(K) = \{\beta(p_i, q_i)\}_{i=1}^n$. It is evident that if C and D are $\bigoplus$-orthogonal finite sequences of elements from P and Q, then $\beta(C \otimes D)$ is $\bigoplus$-orthogonal in L and $\beta(\bigoplus C, \bigoplus D) = \bigoplus \beta(C \otimes D)$.

7. Tensor products

In the present section, we define a tensor product of difference posets and a necessary and sufficient condition for it to exist.

Definition 7.1. *Let P and Q be difference posets. We say that a pair (T, τ) consisting of a difference poset T and a bimorphism $\tau : P \times Q \to T$ is a tensor product of P and Q iff the following conditions are satisfied:*

(i) *If L is a D-poset and $\beta : P \times Q \to L$ is a bimorphism, there exists a morphism $\phi : T \to L$ such that $\beta = \phi \circ \tau$.*

(ii) *Every element of T is a finite orthogonal sum of elements of the form $\tau(p, q)$ with $p \in P$, $q \in Q$.*

It is not hard to show that if a tensor product (T, τ) of P and Q exists, it is unique up to an isomorphism, i.e., if (T, τ) and (T^*, τ^*) are tensor products of D-posets P and Q, then there is a unique isomorphism $\phi : T \to T^*$ such that $\phi(\tau(p, q)) = \tau^*(p, q)$ for all $p \in P$, $q \in Q$.

Now we present the main assertion of this section.

Theorem 7.2. *The difference posets P and Q admit a tensor product if and only if there is at least one difference poset L for which there is a bimorphism $\beta : P \times Q \to L$.*

Proof. The necessary condition is evident.

For sufficiency, suppose that N is the subset of $P \times Q$ consisting of all (p, q) such that $\beta(p, q) = 0$ for every bimorphism β on $P \times Q$. Define $X := (P \times Q) \setminus N$. If $A = \{(p_i, q_i)\}_{i=1}^n$ is a finite sequence of elements from $P \times Q$ and $\beta : P \times Q \to L$ is a bimorphism, it is clear that $\beta(A)$ is $\bigoplus$-orthogonal iff $\beta(\tilde{A})$ is $\bigoplus$-orthogonal, where $\tilde{A} = \{(p_i, q_i)\}_{i=1}^m$, $0 \leq m \leq n$, and $(p_i, q_i) \in A$, $(p_i, q_i) \in X$; in this case $\bigoplus \beta(A) = \bigoplus \beta(\tilde{A})$ for every bimorphism β on $P \times Q$.

Denote by $\mathscr{H}$ the set of all finite sequences H of elements from X such that for every bimorphism β, $\beta(H)$ is a finite decomposition of 1. It is clear that $\mathscr{H}$ is nonempty, since $\{(1, 1)\} \in \mathscr{H}$.

Let $\mathscr{E}(\mathscr{H})$ be the set of all finite sequences $A = \{(p_i, q_i)\}_{i=1}^n$ (may be also empty) such that there is a system $\{(a_j, b_j)\}_{j=1}^m$ of elements from X such that $\{(p_1, q_1), \dots, (p_n, q_n), (a_1, b_1), \dots, (a_m, b_m)\} \in \mathscr{H}$.

On $\mathscr{E}(\mathscr{H})$ we define a relation $\sim$ such that $A \sim B$ iff $\bigoplus \beta(A) = \bigoplus \beta(B)$ for every bimorphism β on $P \times Q$ (if $A = \varnothing$, we put $\bigoplus \beta(\varnothing) := 0$). Then $\sim$ is an equivalence relation, and we let $\pi(A) = \{B \in \mathscr{E}(\mathscr{H}) : B \sim A\}$.

Organize $\Pi(X) := \{\pi(A) : A \in \mathscr{E}(\mathscr{H})\}$ into a difference poset as follows. We say that $\pi(A) \leq \pi(B)$, where

$$A = \{(p_1, q_1), \dots, (p_n, q_n)\}, \quad B = \{(r_1, s_1), \dots, (r_m, s_m)\}, \tag{7.1}$$

iff there is

$$C = \{(p'_1, q'_1), \dots, (p'_s, q'_s)\} \in \mathscr{E}(\mathscr{H}) \tag{7.2}$$

such that

$$M := \{(p_1, q_1), \dots, (p_n, q_n), (p'_1, q'_1), \dots, (p'_s, q'_s)\} \in \mathscr{E}(\mathscr{H}) \tag{7.3}$$

and

$$\bigoplus \beta(M) = \bigoplus \beta(B) \tag{7.4}$$

for every bimorphism β on $P \times Q$. It is straightforward to verify that $\leq$ is a partial ordering on $\Pi(X)$ and $\pi(\varnothing)$ and $\pi(H)$, where H is any element of $\mathscr{H}$, are the smallest and greatest elements of $\Pi(X)$.

The difference $\ominus$ is defined on $\Pi(X)$ via $\pi(B) \ominus \pi(A) = \pi(C)$ iff $\pi(A) \leq \pi(B)$, and A, B, C satisfy the properties (7.1)–(7.4). Verifying conditions of Remark 2.2, we can prove that $\Pi(X)$ is a difference poset. Evidently that if $A = \{(p_1, q_1), \dots, (p_n, q_n)\}$, $B = \{(r_1, s_1), \dots, (r_m, s_m)\} \in \mathscr{E}(\mathscr{H})$, then (i) $\pi(A)^\perp = \pi(A')$, where $A' = \{(u_1, v_1), \dots, (u_t, v_t)\} \in \mathscr{E}(\mathscr{H})$ and $\{(p_1, q_1), \dots, (p_n, q_n), (u_1, v_1), \dots, (u_t, v_t)\} \in \mathscr{H}$; (ii) $\pi(A) \perp \pi(B)$ iff there is $C \in \mathscr{E}(\mathscr{H})$ such that $\bigoplus \beta(A) \oplus \bigoplus \beta(B) = \bigoplus \beta(C)$ for each bimorphism β on $P \times Q$, and then $\pi(A) \oplus \pi(B) = \pi(C)$.

Now put $P \otimes Q := \Pi(X)$ and define a mapping $\otimes : P \times Q \to P \otimes Q$ via

$$\otimes(p, q) = \begin{cases} \pi(\{(p, q)\}), & (p, q) \in X, \\ 0, & (p, q) \notin X. \end{cases} \tag{7.5}$$

For simplicity, we often write $p \otimes q$ rather than $\otimes(p, q)$.

We assert that $\otimes : P \times Q \to P \otimes Q$ is a bimorphism. Indeed, since $\{(1, 1)\} \in \mathscr{H}$, we have $\otimes(1, 1) = \pi(\{(1, 1)\}) = 1$. Suppose that $a, b \in P$ with $a \perp b$ and $q \in Q$. We have to show that $a \otimes q \perp b \otimes q$ and $(a \oplus b) \otimes q = (a \otimes q) \oplus (b \otimes q)$. If $(a, q) \in N$ or $(b, q) \in N$, this is clear, so we may assume that $(a, q), (b, q) \in X$. If β is any bimorphism on $P \times Q$, we have $\beta(a \oplus b, q) = \beta(a, q) \oplus \beta(b, q)$. Hence $\{(a \oplus b, q)\} \sim \{(a, q), (b, q)\}$, so that $(a \oplus b) \otimes q = (a \otimes q) \oplus (b \otimes q)$.

A similar argument shows that $p \otimes (c \oplus d) = (p \otimes c) \oplus (p \otimes d)$ holds for $p \in P$ and $c, d \in Q$ with $c \perp d$.

It remains to prove that $(P \otimes Q, \otimes)$ is a tensor product of P and Q. Since every element of $P \otimes Q = \Pi(X)$ can be written in the form $\pi(A) = \bigoplus\{\pi(\{(p, q)\}) : (p, q) \in A\} = \bigoplus\{p \otimes q : (p, q) \in A\}$, every element of $P \otimes Q$ is a $\bigoplus$-sum of finitely many elements $p \otimes q$.

Finally, suppose that $\beta : P \times Q \to L$ is a bimorphism. If $A, B \in \mathscr{E}(\mathscr{H})$ and $A \sim B$, then $\bigoplus \beta(A) = \bigoplus \beta(B)$; hence we can define a mapping $\phi : P \otimes Q \to L$ by $\phi(\pi(A)) = \bigoplus \beta(A)$ for every $\pi(A) \in \Pi(X)$. Obviously, ϕ is a morphism and we have $\beta(p, q) = \phi(p \otimes q)$ for all $p \in P, q \in Q$. □

Unless confusion threatens, we usually refer to $P \otimes Q$ rather than to $(P \otimes Q, \otimes)$ as being a tensor product.

Corollary 7.3. *The tensor product of the set of all effects* $\mathscr{E}(H_1)$ *and* $\mathscr{E}(H_2)$ *over the same field exists.*

Proof. It follows from Example 6.1 and Theorem 7.2. □

8. Probability measures and tensor products

In this section, we give a sufficient condition for P and Q to admit a tensor product. We show that if the difference posets P and Q have sufficient systems of probability measures, then $P \otimes Q$ exists. For the rest of this section, we use the following notation: $\Lambda := \Omega(P) \times \Omega(Q)$ and, if $\lambda = (\mu, \nu) \in \Lambda$ and $(p, q) \in P \times Q$, then $\lambda(p, q) := \mu(p) \cdot \nu(q)$.

Theorem 8.1. *Let the difference posets P and Q have sufficient systems of probability measures. Then $P \otimes Q$ exists and, for $(\mu, \nu) \in \Lambda$, there is a unique probability measure $\mu \otimes \nu \in \Omega(P \otimes Q)$ such that*

$$\mu \otimes \nu(p \otimes q) = \mu(p)\,\nu(q)$$

holds for all $(p, q) \in P \times Q$.

Proof. Let X be the subset of $P \times Q$ consisting of all pairs (p, q) with $p \neq 0, q \neq 0$. If $M = \{(p_1, q_1), \dots, (p_n, q_n)\}$ is a finite sequence of elements from X and $\lambda \in \Lambda$, we put

$$\lambda(M) = \sum_{i=1}^{n} \lambda(p_i, q_i)$$

with the understanding that if $M = \varnothing$, then $\lambda(M) = 0$.

Now define the set $\mathscr{F}$ of all finite sequences $T = \{(p_i, q_i)\}_{i=1}^n$ of elements in X such that $\lambda(T) = 1$ for any $\lambda \in \Lambda$. Since $\lambda(1, 1) = 1$, $\mathscr{F}$ is nonvoid. It is clear that if $(p, q) \in X$, then from the set $\{(p, q), (p^\perp, q), (p, q^\perp), (p^\perp, q^\perp)\} \cap X$ we can choose a finite sequence containing (p, q) and belonging to $\mathscr{F}$. Denote by $\mathscr{E}(\mathscr{F})$ the set of all finite sequences $\{(p_j, q_j) : j \in J\}$ such that $J \subseteq I$ and $\{(p_i, q_i) : i \in I\} \in \mathscr{F}$. We put $\{(p_i, q_i) : i \in \varnothing\} = 0$.

For two events $A, B \in \mathscr{E}(\mathscr{F})$ we define $A \sim B$ iff $\lambda(A) = \lambda(B)$ for any $\lambda \in \Lambda$. Then $\sim$ is an equivalence on $\mathscr{E}(\mathscr{F})$, and let $\pi(A) := \{B \in \mathscr{E}(\mathscr{F}) : B \sim A\}$. Let $\Pi(X) = \{\pi(A) : A \in \mathscr{E}(\mathscr{F})\}$. We organize $\Pi(X)$ into a poset by defining a partial ordering $\leq$ on $\Pi(X)$ as follows: $\pi(A) \leq \pi(B)$, where $A = \{(p_1, q_1), \dots, (p_n, q_n)\}$, $B = \{(r_1, s_1), \dots, (r_m, s_m)\}$, iff there is $C = \{(p_1', q_1'), \dots, (p_s', q_s')\} \in \mathscr{E}(\mathscr{F})$ such that $M := \{(p_1, q_1), \dots, (p_n, q_n), (p_1', q_1'), \dots, (p_s', q_s')\} \in \mathscr{E}(\mathscr{F})$ and $\lambda(M) = \lambda(B)$ for any $\lambda \in \Lambda$. Then $\pi(\varnothing)$ and $\pi(T)$, where $T \in \mathscr{F}$, are the smallest and greatest elements in $\Pi(X)$.

The difference operation $\ominus$ on $\Pi(X)$ is defined whenever $\pi(A) \leq \pi(B)$, and $\pi(B) \ominus \pi(A) = \pi(C)$, where A, B, C satisfy the above-mentioned conditions for the partial ordering $\leq$. Then $\ominus$ is defined correctly and $\Pi(X)$ is a difference poset.

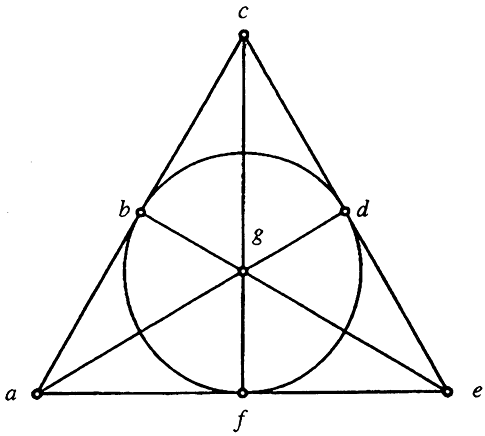

FIGURE 1

Define a mapping $\gamma : P \times Q \to \Pi(X)$ via

$$\gamma(p, q) = \begin{cases} \pi(\{(p, q)\}), & (p, q) \in X, \\ 0, & (p, q) \notin X. \end{cases}$$

Then γ is, evidently, a bimorphism; hence by Theorem 7.2, there is a tensor product $P \otimes Q$ and a morphism $\phi : P \otimes Q \to \Pi(X)$ such that $\gamma(p, q) = \phi(p \otimes q)$ for all $(p, q) \in P \times Q$.

If $\lambda = (\mu, \nu) \in \Lambda$, define $\mu \otimes \nu : P \otimes Q \to [0, 1]$ by $\mu \otimes \nu(t) = \lambda(\phi(t))$ for all $t \in P \otimes Q$. Then $\mu \otimes \nu$ is a well-defined probability measure on $P \otimes Q$ and $\mu \otimes \nu(p \otimes q) = \mu(p)\nu(q)$ for all $(p, q) \in P \times Q$. □

We note that an analogous statement for orthoalgebras has been proved in [7]; however, they assumed that, for any (p, q) with $p \neq 0, q \neq 0$, there is $\lambda \in \Lambda$ such that $\lambda(p, q) > 1/2$.

Corollary 8.2. *Let $\mathscr{E}(H_1)$ and $\mathscr{E}(H_2)$ be two sets of effects in Hilbert spaces H_1 and H_2 (not necessarily over the same field). Then $\mathscr{E}(H_1) \otimes \mathscr{E}(H_2)$ exists.*[2]

Proof. It follows from the facts that for any von Neumann operator T_i on H_i, the mapping

$$\mu_i(A) = \mathrm{tr}(T_i A), \qquad A \in \mathscr{E}(H_i),$$

is a probability measure on $\mathscr{E}(H_i)$ and $\Omega(\mathscr{E}(H_i))$ is therefore sufficient for $i = 1, 2$. The assertion of the corollary now follows from Theorem 8.1. □

Remark 8.3. In [7], it has been shown that the orthoalgebra, the Fano plane, illustrated by the Greechie diagram in Figure 1, has no tensor product $F \otimes F$ in the category of orthoalgebras. We show that the tensor product as a D-poset exists. It follows from the fact that $\Omega(F) = \{\mu\}$, where $\mu(x) = 1/3$, $x \in \{a, b, c, d, e, f, g\}$. Using Theorem 8.1, we see that $F \otimes F$ as a D-poset exists.

9. Bounded Boolean powers and tensor products

A special kind of a tensor product is needed if we wish to describe a coupled system consisting of one quantum system and one classical one. This situation

[2]The same assertion holds if we use the systems of all effects in von Neumann algebras.

arises, for example, by quantum measurements, where we wish to measure a quantum observable by a measuring device [2].

In the rest of the paper, we shall study a tensor product of a difference poset L (i.e., a logic of a quantum system) and a Boolean algebra B (i.e. a logic of a classical system). It is well known that $\Omega(B)$ is sufficient, but on the other hand it can happen that $\Omega(L) = \varnothing$, so that Theorem 8.1 is not effective. Nevertheless we show that the tensor product $L \otimes B$ exists and, in addition, is isomorphic to a bounded Boolean power.

So let L be a difference poset and B a Boolean algebra with the smallest and greatest elements 0_B and 1_B, respectively. According to [4], we define

(9.1)
$$L[B]^* = \left\{ f \in B^L : \ a \neq b \Rightarrow f(a) \wedge f(b) = 0_B,\ f(L) \text{ is finite},\ \bigvee_{a \in L} f(a) = 1_B \right\}.$$

The set $L[B]^*$ is said to be a *bounded Boolean power* of L. In the paper [4], there was also introduced a Boolean power of L for the case when B is a complete Boolean algebra.

Define, for any $a \in L$, a mapping $\hat{a} : L \to B$ via

$$\hat{a}(x) = \begin{cases} 1_B & \text{if } x = a, \\ 0_B & \text{if } x \neq a, \end{cases} \quad x \in L. \tag{9.2}$$

Theorem 9.1. *Let L be a difference poset with $\leq, 0, 1$, and $\ominus$; and let B be a Boolean algebra. For $f, g \in L[B]^*$ we define a partial binary relation $\leq$ via*

$$f \leq g \quad \text{iff} \quad \bigvee_{\substack{x, y \in L \\ x \leq y}} f(x) \wedge g(y) = 1_B, \tag{9.3}$$

and for $f, g \in L[B]^*$ with $f \leq g$ we put $g \ominus f : L \to B$ via

$$(g \ominus f)(x) = \bigvee_{\substack{a, b \in L \\ x = b \ominus a}} g(b) \wedge f(a), \quad x \in L. \tag{9.4}$$

The Boolean power $L[B]^*$ with $\leq, \ominus$ defined via (9.1), (9.3), *and* (9.4) *is a difference poset with the smallest and greatest elements $\hat{0}$ and $\hat{1}$, respectively.*

Proof. It follows the same ideas as that in [4] for Boolean powers, and to illustrate it, we present a typical step of the proof: the antisymmetry of $\leq$.

Now let $f \leq g$, $g \leq f$. Then

$$\begin{aligned} 1_B &= \left(\bigvee_{\substack{x, y \in L \\ x \leq y}} [f(x) \wedge g(y)] \right) \wedge \left(\bigvee_{\substack{u, v \in L \\ u \leq v}} [g(u) \wedge f(v)] \right) \\ &= \bigvee_{\substack{x, y \in L \\ x \leq y}} \bigvee_{\substack{u, v \in L \\ u \leq v}} [f(x) \wedge g(y) \wedge g(u) \wedge f(v)] \\ &= \bigvee_{\substack{x, y, v \in L \\ x \leq y \leq v}} [f(x) \wedge g(y) \wedge f(v)] = \bigvee_{x \in L} [f(x) \wedge g(x)] \end{aligned}$$

so that, for any $y \in L$,

$$f(y) = \left(\bigvee_{x \in L} f(x) \wedge g(x) \right) \wedge f(y)$$
$$= \bigvee_{x \in L} [f(x) \wedge f(y) \wedge g(x)] = f(y) \wedge g(y);$$

hence, $f(y) \leq g(y)$. By symmetry, $g(y) \leq f(y)$; hence, $f(y) = g(y)$, $y \in L$, and $f = g$.

For more details, see [4]. □

Let B be a Boolean algebra and L a difference poset. Let $T = \{t_i : i \in I\}$ be a finite decomposition of 1_B, i.e. $t_i \wedge t_j = 0_B$ if $i \neq j$ and $\bigvee_{i \in I} t_i = 1_B$. If $\{f_i : i \in I\} \subseteq L[B]^*$, then

(9.5)
$$f(x) = \bigvee_{i \in I} (f_i(x) \wedge t_i), \qquad x \in L,$$

is an element of $L[B]^*$. For (9.5) we can use $f = \bigvee_{i \in I} f_i \wedge t_i$ or the "sum" notation

(9.6)
$$f = \sum_{i \in I} f_i \cdot t_i.$$

In particular, if $\{a_i : i \in I\} \subseteq L$ and $\{t_i : i \in I\}$ is a finite decomposition of 1_B, then

(9.7)
$$\sum_{i \in I} \hat{a}_i \cdot t_i$$

belongs to $L[B]^*$. Conversely, any element $f \in L[B]^*$ can be written in the form (9.7) for appropriate pairwise different a_i's in L and a finite decomposition $T = \{t_i : i \in I\}$. Indeed, given $f \in L[B]^*$ we put $I = L$ and $T = \{f(a) : a \in L\}$. Then

$$f = \sum_{a \in L} \hat{a} \cdot t_a,$$

where $t_a = f(a)$, $a \in L$.

In addition, we may assume that the finite decomposition of 1_B is strictly positive (i.e., $t_i \neq 0_B$ for each i). This form is called the *reduced representation* of f by its values. We recall that in this case f has a unique reduced representation. Indeed, if $f = \sum_i \hat{a}_i \cdot t_i = \sum_j \hat{b}_j \cdot s_j$, $t_i, s_j > 0_B$, and $\{a_i\}$ and $\{b_j\}$ consist of pairwise different elements, then $f(x) = 0_B$ iff $x \neq a_i$ for any i and $f(x) = t_i$ iff $x = a_i$, so that $a_i = b_j$ and $t_i = s_j$ for some i and j.

Theorem 9.2. *Let L be a difference poset, and let B be a Boolean algebra. Then the mappings $\lambda : L \to L[B]^*$, defined via $\lambda(a) = \hat{a}$, where $\hat{a}$ is defined via (9.2), and $\beta : B \to L[B]^*$, defined for $b \in B$ via*

(9.8)
$$\beta(b)(x) \begin{cases} b & \text{if } x = 1_L, \\ b^c & \text{if } x = 0_L, \\ 0_B & \text{otherwise}, \end{cases} \quad x \in L,$$

are monomorphisms preserving all existing suprema (infima) in L and $L[B]^*$, respectively. In particular,

$$\lambda\left(\bigoplus_i a_i\right) = \bigoplus_i \lambda(a_i),$$

whenever $\bigoplus_i a_i$ exists in L.

Proof. We recall that the partial binary operation $\oplus$ on $L[B]^*$ can be defined via

$$(f \oplus g)(x) = \bigvee_{\substack{u,v\in L\\ u\oplus v=x}} f(u) \wedge g(v), \qquad x \in L,$$

and $f^\perp(x) = f(x^\perp)$, $x \in L$.

Since $\lambda(a) \le \lambda(b)$ iff $a \le b$, we argue that if $a = \bigvee_i a_i$, then $\lambda(a) \ge \lambda(a_i)$ for any i. If for $g \in L[B]^*$ we have $g \ge \lambda(a_i)$ for any i, we have

$$1_B = \bigvee_{\substack{x,y\in L\\ x\le y}} g(y) \wedge \hat{a}_i(x) = \bigvee_{\substack{y\in L\\ y\ge a_i}} g(y),$$

which gives

$$1_B = \bigvee_{\substack{y\in L\\ y\ge a}} g(y) = \bigvee_{\substack{x,y\in L\\ x\le y}} g(y) \wedge \hat{a}(x),$$

so that $\lambda(a) \le g$.

For β we conclude as follows. We have $\beta(b)(x) = (\hat{1}(x) \wedge b) \vee (\hat{0}(x) \wedge b^c)$ or $\beta(b) = \hat{1}\cdot b + \hat{0}\cdot b^c$. Hence $\beta(1)(x) = \hat{1}(x)$ and $\beta(b^c)(x) = (\hat{1}(x) \wedge b^c) \vee (\hat{0}(x) \wedge b) = \beta(b)(x^\perp)$, since $\hat{1}(x^\perp) = \hat{0}(x)$ and $\hat{0}(x^\perp) = \hat{1}(x)$. Let $f \in L[B]^*$. Then

$$\beta(b)(x) \wedge f(y) = \begin{cases} b \wedge f(y) & \text{if } x = 1_L, \\ b^c \wedge f(y) & \text{if } x = 0_L, \\ 0_B & \text{if } x \ne 1_L,\ x \ne 0_L; \end{cases}$$

and therefore

$$\bigvee_{\substack{x,y\in L\\ x\le y}} [\beta(b)(x) \wedge f(y)] = (b \wedge f(1)) \vee \left(\bigvee_{y\in L} [b^c \wedge f(y)]\right) = b^c \vee (b \wedge f(1)),$$

which entails that $\beta(b) \le f$ if and only if $b \le f(1)$. For $b_1, b_2 \in B$ this gives that $\beta(b_1) \le \beta(b_2)$ if and only if $b_1 \le b_2$. Now assume that $b = \bigvee_i b_i$. Let f be any upper bound of $\beta(b_i)$, for any i. From $b_i \le f(1)$, for any i we get $\bigvee_i b_i \le f(1)$; hence $\beta(b) \le f$. That is, $\beta(b) = \bigvee_i \beta(b_i)$. This proves that $\beta : B \to L[B]^*$ is an embedding.

Theorem 9.3. *Let L be a difference poset and B a Boolean algebra. Then the tensor product $L \otimes B$ exists and is isomorphic to the bounded Boolean power $L[B]^*$.*

Proof. Define a mapping $\gamma_o : L \times B \to L[B]^*$ as

(9.9) $$\gamma_o(a, b) = \hat{a}\cdot b + \hat{0}\cdot b^c, \qquad (a, b) \in L \times B.$$

Then from Theorem 9.2 we conclude that γ_o is a bimorphism on $L \times B$ and, by Theorem 7.2, $L \otimes B$ exists. Therefore, there is a morphism $\gamma : L \otimes B \to L[B]^*$ such that $\gamma(a \otimes b) = \gamma_o(a, b)$. If we use a reduced representation $f = \sum_{i=1}^n a_i \cdot t_i = \bigoplus_{i=1}^n \gamma_o(a_i, t_i)$, we see from

(i) $\gamma_o(a_1, b_1) \perp \gamma_o(a_2, b_2)$ iff $a_1 \perp a_2$ or $b_1 \perp b_2$;
(ii) $t = \bigoplus_{i=1}^n a_i \otimes t_i \in L \otimes B$ and $\gamma(t) = f$,

that γ is surjective onto $L[B]^*$. Since the reduced representation is unique, we conclude that γ is a monomorphism, and so γ is an isomorphism, which proves that $L[B]^*$ is isomorphic with $L \otimes B$. □

Let, for any $i \in I$, L_i with $\leq_i$, 1_i, $\ominus_i$ be a D-poset. Then $L := \prod_{i \in I} L_i$ is a D-poset, called a *product D-poset* of $\{L_i : i \in I\}$, when $\leq$, 1, and $\ominus$ are defined on L as follows: $\{a_i\} \leq \{b_i\}$ iff $a_i \leq_i b_i$, $i \in I$, $1 = \{1_i\}$, $\{b_i\} \ominus \{a_i\} = \{b_i \ominus_i a_i\}$.

Theorem 9.4. *Let $B = 2^n$, and let L be a difference poset. Put $L_o = \prod_{j=1}^n L_j$, where $L_j = L$ for $j = 1, \ldots, n$. Then L_o, $L[B]^*$, and $L \otimes B$ are isomorphic.*

Proof. We can assume that $N = \{1, \ldots, n\}$ and $B = 2^N$. Then $\{\{j\} : j = 1, \ldots, n\}$ are atoms of B. For $\{a_j\}_{j=1}^n \in L_o$ we define an element of $L[B]^*$ via $\sum_{j=1}^n \hat{a}_j \cdot \{j\}$. It is not hard to show that the mapping $h : L_o \to L[B]^*$ such that $\{a_j\}_{j=1}^n \mapsto \sum_{j=1}^n \hat{a}_j \cdot \{j\}$ is a monomorphism from L_o into $L[B]^*$.

On the other hand, let $f \in L[B]^*$ and let it have the reduced representation $f = \sum_i \hat{a}_i \cdot t_i$. Then, for any $x \in L$, we have

$$\begin{aligned} f(x) &= \bigvee_i \hat{a}_i(x) \wedge t_i = \bigvee_i \bigvee_{j=1}^n \hat{a}_i(x) \wedge t_i \wedge \{j\} \\ &= \bigvee_{j=1}^n \bigvee_{\substack{i \\ \{j\} \leq t_i}} \hat{a}_i(x) \wedge t_i \wedge \{j\} \wedge \bigvee_{j=1}^n \bigvee_{\substack{i \\ \{j\} \perp t_i}} \hat{a}_i(x) \wedge t_i \wedge \{j\} \\ &= \bigvee_{j=1}^n \bigvee_{\substack{i \\ \{j\} \leq t_i}} \hat{a}_i(x) \wedge t_i \wedge \{j\}. \end{aligned}$$

If we put $c_j = a_i$ whenever $\{j\} \leq t_i$, then $f = \sum_{j=1}^n \hat{c}_j \cdot \{j\}$ which proves that h is surjective. □

Finally, we recall that if L is a Boolean algebra, so is $L[B]^*$, and there exists a tensor product of Boolean algebras which in view of Theorem 9.3 is always a Boolean algebra.

10. Concluding remarks

In the paper, we defined a tensor product of difference posets via a universal mapping property on the class of difference posets. We presented also an equivalent condition, Theorem 7.2, and we proved that a tensor product of difference posets with sufficient systems of probability measures always exists, Theorem 8.1. Moreover, a D-poset and a Boolean algebra admit a tensor product which, in addition, is isomorphic to a bounded Boolean power, Theorem 9.3.

We recall that the problem of whether any two D-posets admit a tensor problem seems to be open.

We note that the presented proofs have used some ideas developed by Foulis and Bennett in [7]; however, their main tool, an algebraic test space, is not effective in the case of D-posets because it leads to orthoalgebras. Our method enables us to prove more general statements as those in [7], and we hope to develop it in the future because of its useful applications in quantum measurement modeling.

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